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Two-Dimensional Subsonic Flow of Compressible Fluids

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SUMMARY

The basic concept of the present paper is to use a tangent line to the adiabatic pressure-volume curve as an approximation to the curve itself. First, the general characteristics of such a fluid are shown. Then in Section I a theory is developed which can be applied to flows with velocities approaching that of sound, whereas the theory of Demtchenko and Busemann only give an approximation for flows with velocities smaller than one-half of the sound velocity. This is done by a generalization of the method of approximation to the adiabatic relation by a tangent line, conceived jointly by Th. von Kármán and the author. The theory is put into a form by which, knowing the incompressible flow over a body, the compressible flow over a similar body can be calculated. The theory is then applied to calculate the flow over elliptic cylinders. In Section II the work of H. Bateman is applied to this approximate adiabatic fluid and the results obtained are essentially the same as those obtained in Section I.

INTRODUCTION

Assuming that the pressure is a single-valued function of density only, the equations of two-dimensional irrotational motion of compressible fluids can be reduced to a single non-linear equation of the velocity potential. In the supersonic case, that is, in the case when flow velocity is everywhere greater than that of local sound velocity, the problem is solved by Meyer and Prandtl and Busemann using the method of characteristics. The essential difficulty of this problem lies in the subsonic case, that is, in the case when flow velocity is everywhere smaller than but near the local sound velocity, because then the method of characteristics cannot be used. Glauert and Prandtl[1] treated the case when the disturbance of parallel rectilinear flow, due to the presence of a solid body, is small. They were then able to linearize the differential equation for the velocity potential and obtain an equation very similar to that for incompressible fluids. But there are usually stagnation points either on the surface of the body or in the field of flow, where the disturbance is no longer small. Hence, it is doubtful whether the linear theory can be applied to the flow near a stagnation point. For the same reason, the theory breaks down in the case of bodies whose dimension across the stream is not small compared with the dimension parallel to the stream.

To treat these cases Janzen and Rayleigh developed the method of successive approximations. This method was put into a more convenient form by Poggi and Walther. Recently, Kaplan[2] treated the case of flow over Joukowsky air-foils and elliptic cylinders using Poggi's method. However, the method is rather tedious and the convergence very slow if the local velocity of sound is approached.

Molenbroek and Tschapligin suggested the use of the magnitude of velocity w and inclination β of velocity to a chosen axis as independent variables, and were thus able to reduce the equation of velocity potential to a linear equation. This equation was solved by Tschapligin. The solution is essentially a series, each term of which is a product of a hypergeometric function of w and a trigonometric function of β. The main difficulty in practical application of this solution is to obtain a proper set of boundary conditions in the plane of independent variables w, β and to put the solution in a closed form.

Tschapligin has shown that a great simplification of the equation in the hodograph plane results if the ratio of the specific heats of the gas is equal to -1. Since all real gases have their ratio of specific heats between 1 and 2, the value -1 seems without practical significance. Demtchenko[3] and Busemann[4] clarified the meaning of this specific value of -1. They found that this really means to take the tangent of the pressure-volume curve as an approximation to the curve itself. However, they limit themselves to the use of the tangent at the state of the gas corresponding to the stagnation point of flow. As a result their theory can only be applied to a flow with velocities up to about one-half the velocity of sound. Recently, during a discussion with Th. von Kármán he suggested to the author that the theory can be generalized by using the tangent at the state of the gas corresponding to undisturbed parallel flow. Thus the range of usefulness of the theory can be greatly extended. This is carried out in the first section of the present paper. This theory, based upon Demtchenko and Busemann's work, is then applied to the case of flow over elliptic cylinders and the results compared with those of Hooker[5] and Kaplan.[2] Furthermore, results calculated by Glauert-Prandtl linear theory are also included for comparison.

Recently, Bateman[6] demonstrated a remarkable reciprocity between two fields of flow, of two fluids related by a certain point transformation. It is shown in the second section of this paper that the flow of an incompressible fluid and the flow of a compressible fluid approximated by the use of the tangent to adiabatic pressure-volume curve can be interpreted as such a

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point transformation. It is therefore possible to obtain a solution for compressible flow whenever a solution of incompressible flow is known. This transformation from the flow of an incompressible fluid to the flow of a compressible fluid is found, however, to be essentially the same as that developed from Demtchenko and Busemann's work.

Approximation to the Adiabatic Relation

If p is the pressure, v is the specific volume and γ is the ratio of specific heats of a gas, the adiabatic relation pv^γ = constant is a curve in the p-v plane as shown in Fig. 1(a). The conditions near the point p_1, v_1 which

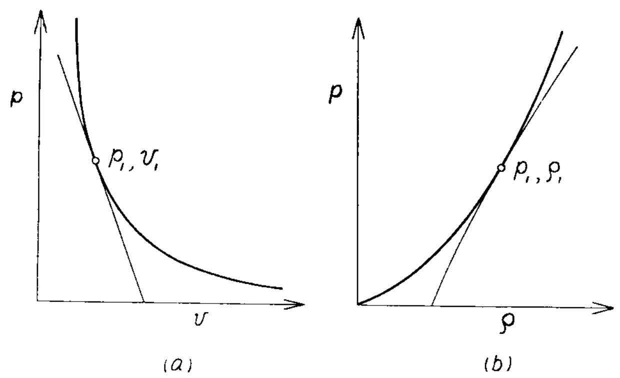


FIG. 1. Approximation to the adiabatic p-v relation by means of a tangent.

correspond to a state of undisturbed flow can be approximated by the tangent to the curve at that point. The equation of the tangent at this point can be written

$$p_1 - p = C(v_1 - v) = C(\rho_1^{-1} - \rho^{-1}) \tag{1}$$

where C is the slope of the tangent and ρ is the density of the fluid. The slope C must be equal to the slope of the curve at the point p_1, v_1. Therefore,

$$C = \left(\frac{dp}{dv}\right)_1 = \left(\frac{dp}{d\rho}\frac{d\rho}{dv}\right)_1 = -\left(\frac{dp}{d\rho}\right)_1 \rho_1^2 = -a_1^2\rho_1^2$$

where a_1 is the sound velocity corresponding to the conditions p_1, v_1. Thus Eq. (1) can be written

$$p_1 - p = a_1^2\rho_1^2(\rho^{-1} - \rho_1^{-1}) \tag{2}$$

This is an approximation to the true adiabatic pressure-density relation, and is shown in Fig. 1(b) together with the true adiabatic relation.

The generalized Bernoulli theorem for compressible fluids is:

$$w_2^2 - w_3^2 = 2\int_2^3 dp/\rho \tag{3}$$

where w is the velocity of the gas and the subscripts 2 and 3 denote two different states of the fluid. By substituting Eq. (2) into Eq. (3),

$$w_2^2 - w_3^2 = a_1^2\rho_1^2(\rho_2^{-2} - \rho_3^{-2}) \tag{4}$$

Now if $w_3 = 0$, $w_2 = w$, $\rho_3 = \rho_0$, and $\rho_2 = \rho$, with the subscript 0 denoting the state of the fluid corresponding to the stagnation point of flow, Eq. (4) gives

$$\frac{a_1^2\rho_1^2}{\rho_0^2} + w^2 = \frac{\rho_1^2 a_1^2}{\rho^2} \tag{5}$$

If the square of sound velocity a^2 is, as usual, defined as the derivative of p with respect to ρ, Eq. (2) gives

$$a^2\rho^2 = \rho^2 dp/d\rho = a_1^2\rho_1^2 = \text{constant} \tag{6}$$

Therefore, Eq. (5) can be written as:

$$(\rho/\rho_0)^2 = 1 - (w/a)^2 \tag{7}$$

Similarly,

$$(\rho_0/\rho)^2 = 1 + (w/a_0)^2 \tag{8}$$

It is interesting to note that from Eq. (8) the density decreases as the velocity increases, as may be expected. Therefore Eq. (6) indicates that the local velocity of sound increases as the velocity increases. This behavior is opposite to that of a real gas, since in the case of an adiabatic flow of a real gas it is well known that the temperature of the gas decreases as the velocity increases; thus the local sound velocity, being proportional to the square root of the temperature, also decreases. However, in the present approximate theory, the ratio w/a or the Mach Number, still increases as the velocity increases, as can be seen from Eq. (7). But this ratio only reaches the value unity when $\rho = 0$, or from Eq. (8), when $w = \infty$. It is thus seen that the entire regime of flow is subsonic and the differential equation of the velocity potential is always of an elliptic type, *i.e.*, always of the same type as the differential equation of the velocity potential of incompressible fluids. This is the reason why the complex representation of the velocity potential and the stream function is possible for all cases, as will be shown in the following paragraphs. However, one should realize that the portion of the tangent that could be used as an approximation to the true adiabatic relation is that portion which lies in the first quadrant. Thus the upper velocity limit for practical application of the theory occurs at $p = 0$. By using Eqs. (7) and (8) this upper limit is found to be

$$\left(\frac{w}{w_1}\right)_{max.} = \frac{1}{(w_1/a_1)}\sqrt{\left(\frac{p_1}{a_1^2\rho_1} + 1\right)^2 - \left[1 - \left(\frac{w_1}{a_1}\right)^2\right]} \tag{9}$$

Since the point p_1, ρ_1 being the tangent point to the true adiabatic curve, lies on the curve, the relation $a_1^2 = \gamma p_1/\rho_1$ which is true for the adiabatic relation $p\rho^{-\gamma}$ = constant can be used, and Eq. (9) becomes

$$\left(\frac{w}{w_1}\right)_{max.} = \frac{1}{(w_1/a_1)}\sqrt{\left(\frac{1}{\gamma} + 1\right)^2 - \left[1 - \left(\frac{w_1}{a_1}\right)^2\right]} \tag{10}$$

This relation is plotted in Fig. 2 with $\gamma = 1.405$. Since for most practical cases it is not likely that the ratio (w/w_1) will rise to values much higher than 2, p will remain positive, and this theory will be sufficient to give an approximate solution.

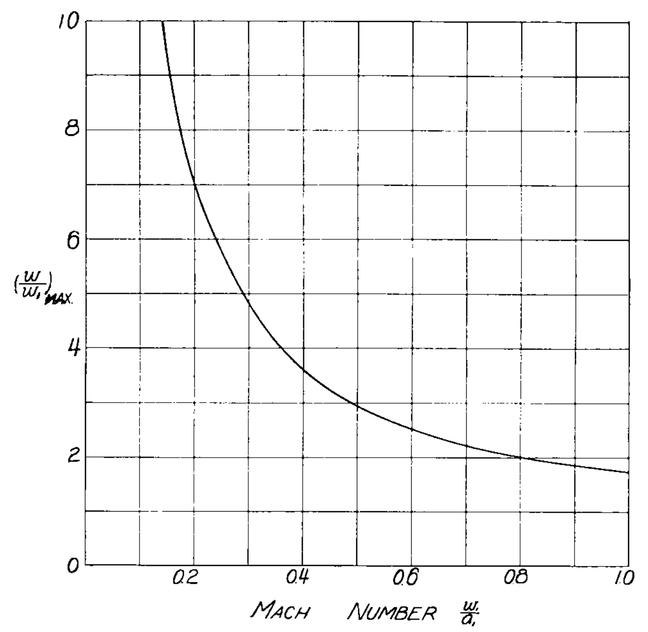


FIG. 2. Relation between the maximum velocity $(w/w_1)_{max}$ (at which the pressure is zero) and the Mach Number (w_1/a_1).

SECTION I

Hodograph Method

If the flow is irrotational, there exists a velocity potential ϕ such that

$$\partial\phi/\partial x = u, \ \partial\phi/\partial y = v \tag{11}$$

where u, v are the components of w in the x and y direction, respectively. The equation of continuity

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}u\right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\rho_0}v\right) = 0$$

will be satisfied, if the stream function ψ is introduced such that:

$$u\rho/\rho_0 = \partial\psi/\partial y, \ -v\rho/\rho_0 = \partial\psi/\partial x \tag{12}$$

If the angle of inclination of the velocity w to the x axis is β, Eqs. (11) and (12) give:

$$\begin{aligned} d\phi &= w\cos\beta dx + w\sin\beta dy \\ d\psi &= -\,w(\rho/\rho_0)\sin\beta dx + w(\rho/\rho_0)\cos\beta dy \end{aligned} \tag{13}$$

Solving for dx and dy,

$$dx = \frac{\cos\beta}{w}d\phi - \frac{\sin\beta}{w}\frac{\rho_0}{\rho}d\psi$$

$$dy = \frac{\sin\beta}{w}d\phi + \frac{\cos\beta}{w}\frac{\rho_0}{\rho}d\psi \tag{14}$$

As long as the correspondence between the physical and hodograph plane is one to one, or mathematically $\partial(x, y)/\partial(u, v) \neq 0$; x and y can be expressed as functions of w, β, and ϕ and ψ as function of w, β. Thus,

$$d\phi = \phi_w'dw + \phi_\beta'd\beta$$

$$d\psi = \psi_w'dw + \psi_\beta'd\beta \tag{15}$$

where primes indicate the derivative with respect to the independent variables indicated as subscripts. Substituting Eq. (15) into Eq. (14), the following expressions for dx and dy are obtained:

$$dx = \left(\frac{\cos\beta}{w}\phi_w' - \frac{\sin\beta}{w}\frac{\rho_0}{\rho}\psi_w'\right)dw + \left(\frac{\cos\beta}{w}\phi_\beta' - \frac{\sin\beta}{w}\frac{\rho_0}{\rho}\psi_\beta'\right)d\beta$$

$$dy = \left(\frac{\sin\beta}{w}\phi_w' + \frac{\cos\beta}{w}\frac{\rho_0}{\rho}\psi_w'\right)dw + \left(\frac{\sin\beta}{w}\phi_\beta' + \frac{\cos\beta}{w}\frac{\rho_0}{\rho}\psi_\beta'\right)d\beta \tag{16}$$

Since the left-hand side of Eqs. (16) are exact differentials, the reciprocity relation can be applied, and, therefore,

$$\frac{\partial}{\partial\beta}\left(\frac{\cos\beta}{w}\phi_w' - \frac{\sin\beta}{w}\frac{\rho_0}{\rho}\psi_w'\right) = \frac{\partial}{\partial w}\left(\frac{\cos\beta}{w}\phi_\beta' - \frac{\sin\beta}{w}\frac{\rho_0}{\rho}\psi_\beta'\right)$$

$$\frac{\partial}{\partial\beta}\left(\frac{\sin\beta}{w}\phi_w' + \frac{\cos\beta}{w}\frac{\rho_0}{\rho}\psi_w'\right) = \frac{\partial}{\partial w}\left(\frac{\sin\beta}{w}\phi_\beta' + \frac{\cos\beta}{w}\frac{\rho_0}{\rho}\psi_\beta'\right) \tag{17}$$

Carrying out these differentiations and simplifying with the aid of Eq. (7), Eq. (17),

$$-\frac{\sin\beta}{w}\phi_w' - \frac{\cos\beta}{w}\frac{\rho_0}{\rho}\psi_w' = -\frac{\cos\beta}{w^2}\phi_\beta' + \frac{\sin\beta}{w^2}\frac{\rho}{\rho_0}\psi_\beta'$$

$$\frac{\cos\beta}{w}\phi_w' - \frac{\sin\beta}{w}\frac{\rho_0}{\rho}\psi_w' = -\frac{\sin\beta}{w^2}\phi_\beta' - \frac{\cos\beta}{w^2}\frac{\rho}{\rho_0}\psi_\beta' \tag{18}$$

Solving for ϕ_w' and ψ_β',

$$\phi_w' = -\frac{\rho}{\rho_0}\frac{1}{w}\psi_\beta'$$

$$\phi_\beta' = \frac{\rho_0}{\rho}w\,\psi_w' \tag{19}$$

Eq. (19) can be further simplified by introducing a new variable ω, such that

$$d\omega = \frac{\rho}{\rho_0}\frac{dw}{w} \tag{20}$$

Then Eq. (19) becomes

$$\phi_\omega' = -\psi_\beta', \ \phi_\beta' = \psi_\omega' \tag{21}$$

This can be easily recognized as the Cauchy-Riemann differential equation, and thus $\phi + i\psi$ must be an analytic function of $\omega - i\beta$. However, for convenience of calculation, another new set of independent variables

$U = W\cos\beta$, $V = W\sin\beta$ are introduced where $W = a_0 e^{\omega}$. Then Eq. (21) can be written as

$$\partial\phi/\partial U = \partial\psi/\partial(-V)$$
$$\partial\phi/\partial(-V) = -\partial\psi/\partial U \quad (22)$$

Integrating Eq. (20),

$$W = 2a_0 w/(\sqrt{a_0^2 + w^2} + a_0) \quad (23)$$

and

$$w = 4a_0^2 W/(4a_0^2 - W^2) \quad (24)$$

Substituting into Eq. (8), the following expression for the density ratio ρ_0/ρ is obtained:

$$\rho_0/\rho = (4a_0^2 + W^2)/(4a_0^2 - W^2) \quad (25)$$

Eqs. (22), (23), (24), and (25) are the basic equations of the present theory. Eq. (22) is the Cauchy-Riemann differential equation, and thus the complex potential $F = \phi + i\psi$ must be an analytic function of $\overline{W} = U - iV$, or:

$$\phi + i\psi = F(U - iV) = F(\overline{W})$$
$$\phi - i\psi = \overline{F}(U + iV) = \overline{F}(W) \quad (26)$$

In Eq. (26), $\overline{W}$ and $\overline{F}$ are the complex conjugates of W and F, respectively.

It is now necessary to find the values of x and y corresponding to a given set of values of U and V, *i.e.*, to find the transformation from the hodograph plane to the physical plane. By using Eqs. (24) and (25), Eq. (14) can be written

$$dx = \frac{U \cdot d\phi}{W^2}\left(1 - \frac{W^2}{4a_0^2}\right) - \frac{V \cdot d\psi}{W^2}\left(1 + \frac{W^2}{4a_0^2}\right)$$
$$dy = \frac{V \cdot d\phi}{W^2}\left(1 - \frac{W^2}{4a_0^2}\right) + \frac{U \cdot d\psi}{W^2}\left(1 + \frac{W^2}{4a_0^2}\right) \quad (27)$$

where $W^2 = U^2 + V^2$. These equations can be combined into one equation by means of Eq. (26). Thus,

$$dz = dx + i dy = \frac{dF}{\overline{W}} - \frac{W \cdot d\overline{F}}{4a_0^2} \quad (28)$$

Hence, if an analytic function $F(\overline{W})$ is given for each value of W, the corresponding real velocity w can be calculated by Eq. (24). Then the coordinate of the point in the physical plane at which this velocity occurs can be calculated by integrating Eq. (28). The pressure at this point is then given by Eq. (2). However, using this procedure, it is not possible to predict whether the chosen function $F(\overline{W})$ will give the desired shape of the solid boundary and flow pattern. In other words, this procedure, in common with all hodograph methods, still suffers the difficulty of boundary conditions.

Transformation from Incompressible Flow to Compressible Flow

However, using the simple relation of Eq. (28) the resulting shape of the body can be ascertained approximately by starting with the function:

$$F(\overline{W}) = \phi + i\psi = W_1 G(\zeta) \quad (29)$$

where W_1 is the transformed undisturbed velocity to be interpreted by Eq. (23), and ζ is the complex coordinate $\xi + i\eta$. This function is so chosen as to give the flow of an incompressible fluid over the desired body shape in coordinates ξ and η. The real velocity in the ζ plane of the incompressible fluid is interpreted as the transformed velocity W in the hodograph plane for the compressible fluid. It is known that

$$\overline{W} = W_1 dG(\zeta)/d\zeta \quad (30)$$

Thus,

$$W = W_1 d\overline{G}(\overline{\zeta})/d\overline{\zeta} \quad (31)$$

where $\overline{G}$ and $\overline{\zeta}$ are the complex conjugates of G and ζ, respectively. With Eqs. (30) and (31), Eq. (28) gives

$$dz = d\zeta - \lambda[d\overline{G}/d\overline{\zeta}]^2 d\overline{\zeta}$$

where $\lambda = {}^1/_4(W_1/a_0)^2$. Integrating,

$$z = \zeta - \lambda \int [d\overline{G}/d\overline{\zeta}]^2 d\overline{\zeta} \quad (32)$$

Therefore, the complex coordinate in the physical plane of the compressible fluid is equal to the corresponding complex coordinate in the physical plane of the incompressible fluid plus a correction term. Since this correction term is usually small, the resulting shape of the body will be quite similar to the one in the incompressible fluid. The factor λ in the correction term depends upon the Mach Number of the undisturbed flow only. This can be shown by means of Eqs. (7), (8), and (23), because from those equations the following relation is obtained:

$$\lambda = \frac{1}{4}\left(\frac{W_1}{a_0}\right)^2 = \frac{(w_1/a_1)^2}{[1 + \sqrt{1 - (w_1/a_1)^2}]^2} \quad (33)$$

where w_1/a_1 is the Mach Number of the undisturbed flow. The values of λ for different Mach Numbers w_1/a_1 are plotted in Fig. 3.

To calculate the velocity in the physical plane, $\overline{W}$ is first obtained from Eq. (30) and then with Eq. (23):

$$\frac{w}{w_1} = \frac{|W|}{W_1}\frac{1 - \lambda}{1 - \lambda(|W|/W_1)^2} \quad (34)$$

If the pressure coefficient $\tilde{\omega}$ at any point is defined as $\tilde{\omega} = (p - p_0)/({}^1/_2\rho_1 w_1^2$, then by using Eq. (2), the following relation is obtained:

$$\tilde{\omega} = (1 + \lambda)\frac{1 - (|W|/W_1)^2}{1 - \lambda(|W|/W_1)^2} \quad (35)$$

Flow over Elliptic Cylinders

The theory will now be applied to calculate the flow over an elliptic cylinder at zero angle of attack. The

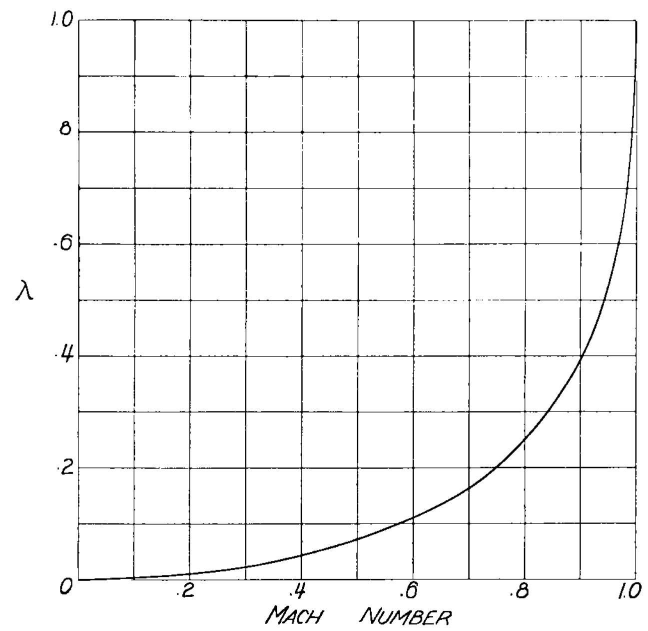


FIG. 3. Variation of the parameter λ of transformation from incompressible flow to compressible flow with the Mach Number w_1/a_1.

incompressible flow over an elliptic cylinder in the complex coordinate ζ can be obtained by applying Joukowsky's transformation to the flow over a circular cylinder in the complex coordinate ζ' with the center of the circle located at the origin of the ζ' plane. Therefore, the function $F(\overline{W})$ or $W_1G(\zeta)$ can be written as:

$$\begin{aligned} F &= W_1(\zeta' + b^2/\zeta) \\ \bar{F} &= W_1(\bar{\zeta}' + b^2/\bar{\zeta}') \end{aligned} \tag{36}$$

where b is the radius of the circle in the ζ' plane. The Joukowsky transformation is:

$$\zeta = \zeta' + 1/\zeta' \tag{37}$$

It is convenient to carry out the computation by using the ζ' coordinates. Thus Eq. (32) is rewritten in the following form:

$$z = \left(\zeta' + \frac{1}{\zeta'}\right) - \lambda \int \left(\frac{d\overline{G}}{d\bar{\zeta}'}\right)^2 \frac{d\bar{\zeta}'}{d\bar{\zeta}/d\bar{\zeta}'} \tag{38}$$

If only the conditions over the surface of the elliptic cylinder are concerned, then:

$$\zeta' = be^{i\theta}, \; \bar{\zeta}' = be^{-i\theta} \tag{39}$$

where θ is the argument as shown in Fig. 4 and b is the radius of the circular section in the ζ' plane which determines the thickness ratio of the elliptic section in the ζ plane. Substituting Eqs. (36), (37), and (39) into Eq. (38), and carrying out the integration, the following expressions for the x and y coordinates corresponding to ζ' are obtained by separating the real and imaginary parts:

$$x = \left(b + \frac{1}{b}\right)\cos\theta - \lambda\left[b(1 + b^2)\cos\theta + \frac{(b^2 - 1)^2}{4}\log\frac{(b^2 - 1)^2 + 4b^2\sin^2\theta}{(b^2 + 2b\cos\theta + 1)}\right]$$

$$y = \left(b - \frac{1}{b}\right)\sin\theta + \lambda\left[b(1 - b^2)\sin\theta + \frac{(b^2 - 1)^2}{2}\tan^{-1}\frac{2b\sin\theta}{b^2 - 1}\right] \tag{40}$$

The horizontal and vertical semi-axis of the approximately elliptic section can then be calculated by substituting $\theta = 0$ and $\theta = \pi/2$, respectively, into Eq. (40). The thickness ratio δ is thus obtained as:

$$\delta = \left(\frac{b^2 - 1}{b^2 + 1}\right) \times \frac{1 + \lambda\left[-b^2 + \frac{b(b^2 - 1)}{2}\tan^{-1}\frac{2b}{b^2 - 1}\right]}{1 - \lambda\left[b^2 + \frac{b(b^2 - 1)}{2}\left(\frac{b^2 - 1}{b^2 + 1}\right)\log\left(\frac{b - 1}{b + 1}\right)\right]} \tag{41}$$

For a given thickness ratio and Mach Number for undisturbed flow, the value of λ is first computed by means of Eq. (33), and then Eq. (41) is solved graphically for b.

After b is obtained, the coordinate x and y for each value of θ can be computed by using Eq. (40). It is fortunate that the values of x, y so obtained lie very close to the true elliptic section. Hence, the velocity and the pressure distribution obtained by using Eqs. (34) and (35) are considered as those over the true elliptic sections.

Calculations for two thickness ratios, $\delta = 0.5$ and $\delta = 0.1$, are carried out and the results shown in Figs. 5 and 6, together with those of Kaplan.[2] Hooker's results[5] are very close to those of Kaplan. Computations are also carried out using the more simple theory of Glauert and Prandtl,[1] and the results are included in Figs. 5 and 6 in order to compare with those of Kaplan and the present theory.

The difference between the various theories lies in the assumptions which are made to simplify the calculations. Glauert and Prandtl assumed that the disturbance introduced by the solid body to the parallel flow is small. In other words, they treated the flow over a body of small thickness ratio. On the other hand, Kaplan and Hooker assume that the Mach Number of the undisturbed flow is small, so that terms containing

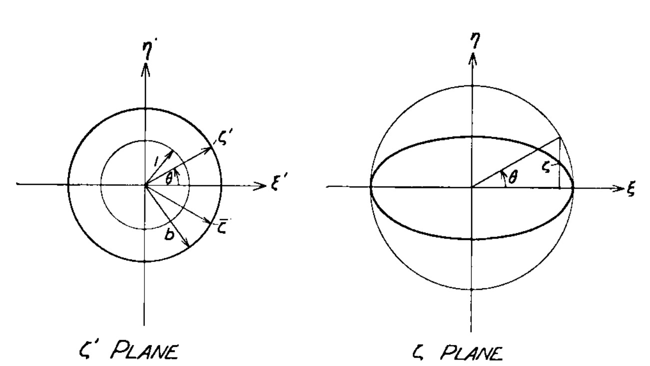


FIG. 4. Notations used in calculating the flow over an elliptical cylinder.

the third and higher powers of the Mach Number can be neglected. The present theory is essentially an improvement of the Glauert-Prandtl theory, so that the effect of large disturbances to the parallel flow is approximately taken into account. Therefore, for flow over thin sections at high Mach Numbers, the result of the present theory should agree well with the Glauert-Prandtl theory, especially at points not too close to the stagnation point. The results of Kaplan and Hooker should show smaller effect of the compressibility due to their second order approximation. For flow over thick sections at lower Mach Numbers, the situation is reversed. In this case results of the present theory should give better agreement with the results obtained by Kaplan and Hooker than with those obtained from the Glauert-Prandtl theory. The above reasoning is substantiated by Figs. 5 and 6.

Critical Velocities for Elliptic Cylinders

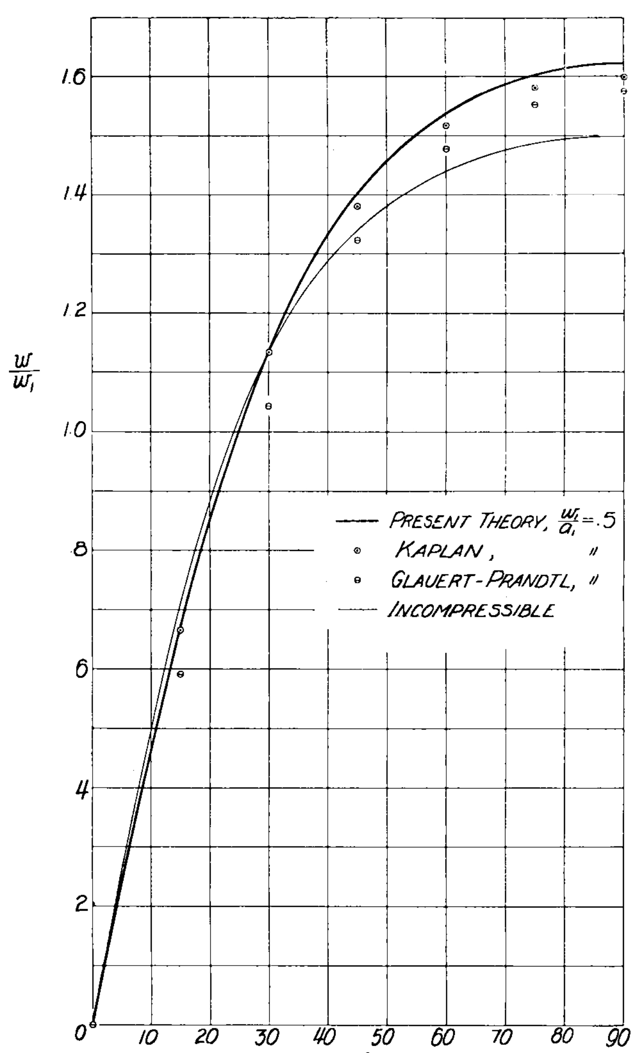
If the velocity of flow over a body is gradually increased, the maximum local velocity in the field will also be increased. When the maximum local velocity reaches the local velocity of sound, shock waves appear and the drag of the body suddenly increases. This velocity is, therefore, of considerable interest to practical engineers and is usually referred to as the critical velocity of the body. It is shown by Kaplan[2] and others that at this critical condition the ratio of maximum velocity of $w_{max.}$ in the field to that of the undisturbed velocity w_1 is related to the Mach Number w_1/a_1 of the undisturbed flow in the following manner:

$$\frac{w_{max.}}{w_1} = \left[\frac{2}{\gamma + 1}\,\frac{1}{(w_1/a_1)^2} + \frac{\gamma - 1}{\gamma + 1}\right]^{1/2} \tag{42}$$

$w_{max.}$ in the flow over an elliptic cylinder at zero angle of attack occurs at the top of the cylinder. Using Eqs. (34) and (33) the value of $w_{max.}/w_1$ is found to be

$$\frac{w}{w_1} = \frac{2b^2/(b^2+1)}{1 + \dfrac{\left[1 - \left(\dfrac{2b^2}{b^2+1}\right)^2\right]\left(\dfrac{w_1}{a_1}\right)^2}{2\left(1 + \sqrt{1 - \left(\dfrac{w_1}{a_1}\right)^2}\right)\sqrt{1 - \left(\dfrac{w_1}{a_1}\right)^2}}} \tag{43}$$

Equating Eqs. (42) and (43) the equation for calculating the critical Mach Number (w_1/a_1) of the undisturbed flow for each value of b is



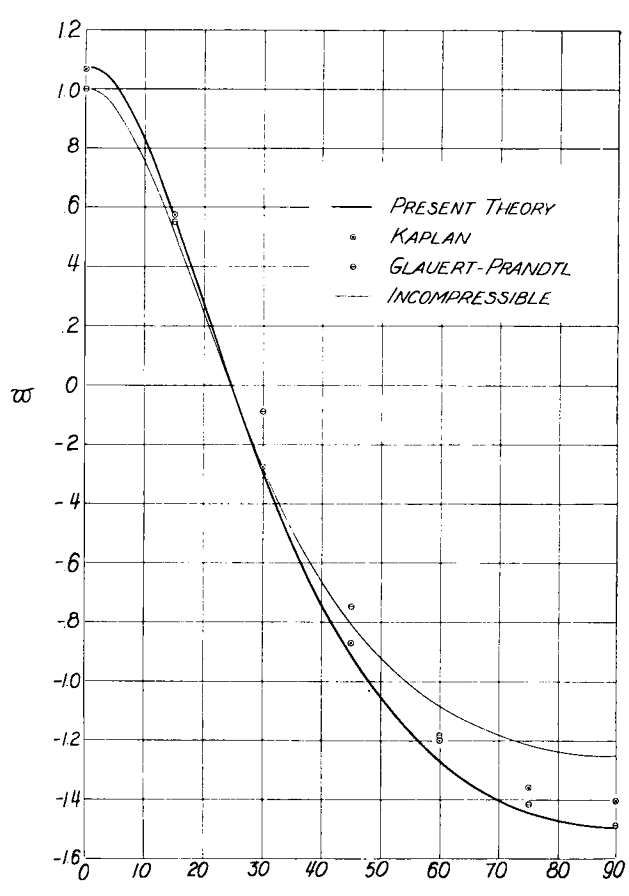


FIG. 5. Flow over an elliptical cylinder with thickness ratio δ = 0.5 at Mach Number = 0.5. (a) Velocity distribution; (b) Pressure distribution.

$$\left[\frac{2}{\gamma+1}\,\frac{1}{(w_1/a_1)_{crit.}}+\frac{\gamma-1}{\gamma+1}\right]^{1/2} = \frac{2b^2/(b^2+1)}{1+\dfrac{\left[1-\left(\dfrac{2b^2}{b^2+1}\right)^2\right]\left(\dfrac{w_1}{a_1}\right)^2_{crit.}}{2\left(1+\sqrt{1-\left(\dfrac{w_1}{a_1}\right)^2_{crit.}}\right)\sqrt{1-\left(\dfrac{w_1}{a_1}\right)^2_{crit.}}}} \quad (44)$$

Knowing $(w_1/a_1)_{crit.}$ for each value of b the corresponding value of δ can be calculated by means of Eqs. (34) and (41). Fig. 7 shows the result of this calculation with Kaplan's value included for comparison. It is seen that the critical Mach Number is lower than that obtained by Kaplan. This lower value of the critical Mach Number indicates a more pronounced effect of compressibility of a fluid and is consistent with the results shown in Figs. 5 and 6.

Section II

The Use of Lift and Drag Functions

If two new functions X and Y are defined by:

$$p_0dX = pdy + \rho_0ud\psi$$
$$p_0dY = \rho_0vd\psi - pdx \quad (45)$$

Assuming that the flow is irrotation, it can be shown by means of Eqs. (11) and (12) that the following relations hold:

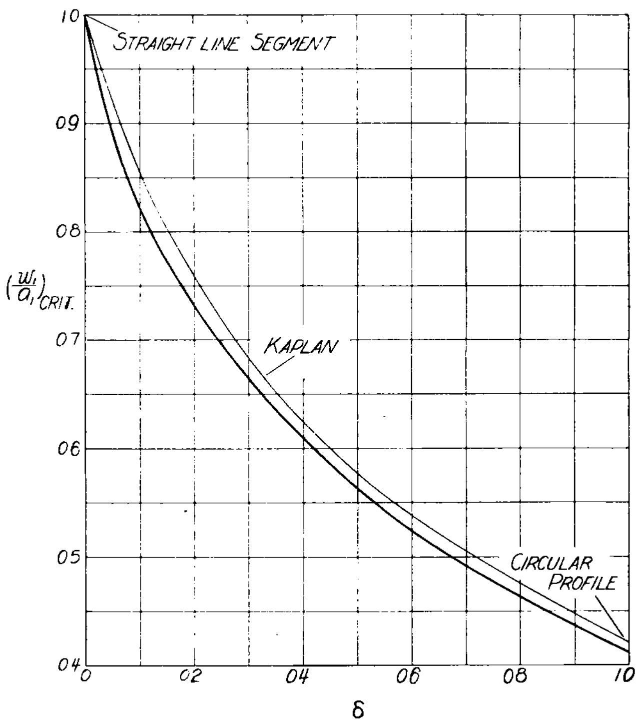
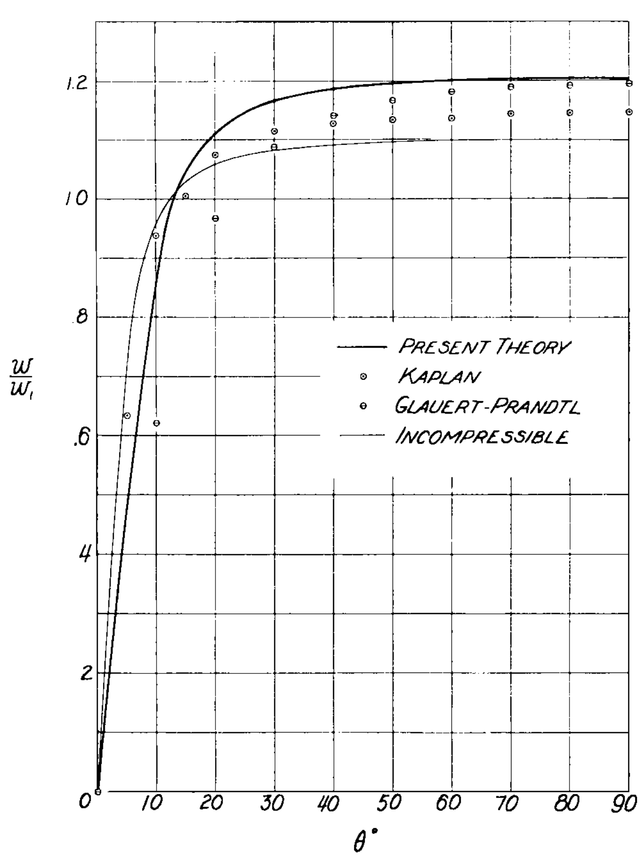


Fig. 7. Variation of critical Mach number $(w_1/a_1)_{crit.}$ of an elliptical cylinder with thickness ratio δ.

$$p_0dX = (p + \rho u^2)dy - \rho uvdx = (p + \rho w^2)dy - \rho vd\phi$$
$$p_0dY = \rho uvdy - (p + \rho v^2)dx = \rho ud\phi - (p + \rho w^2)dx \quad (46)$$

It is seen that by integrating Eq. (46) along any closed boundary, it will give the resultant of the pressure forces acting along the boundary and the rate of increase of momentum of the fluid passing out of the boundary. If there is a solid body in this boundary,



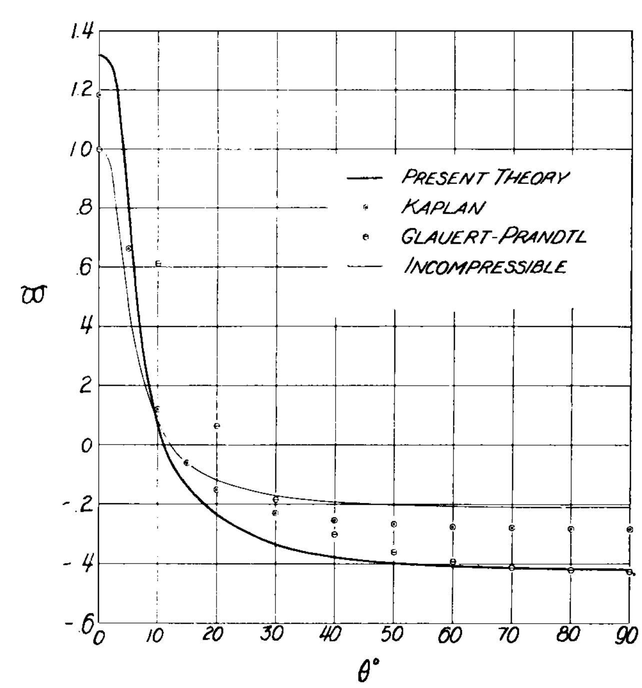


Fig. 6. Flow over an elliptical cylinder with thickness ratio $\delta = 0.1$ at Mach Number $= 0.857$. (a) Velocity distribution; (b) Pressure distribution.

then this integral will give the lift and the drag acting on the body. Therefore, X and Y are sometimes called the drag and lift functions. From Eq. (46) the following relations can be deduced:

$$p_0(vdX - udY) = pd\phi \tag{47}$$

$$p_0(\rho/\rho_0)(udX + vdY) = (p + \rho w^2)d\psi \tag{48}$$

Therefore, by writing $\partial\phi/\partial X = R$ and $\partial\phi/\partial Y = S$, Eq. (47) gives

$$R = \frac{\partial\phi}{\partial X} = \frac{p_0}{p}v, \quad S = \frac{\partial\phi}{\partial Y} = -\frac{p_0}{p}u \tag{49}$$

The quantities R and S have the dimension of a velocity and can be considered as components of a new velocity in the plane whose coordinates are denoted by X and Y. This relation between the xy plane and the XY plane is shown in Fig. 8. It is thus seen that if

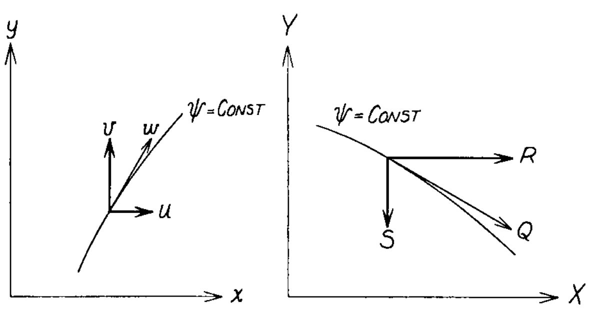


FIG. 8. Relation of the velocity components in the xy plane and XY plane.

the undisturbed flow in the xy plane is in the positive x direction, the undisturbed flow in the XY plane will be in the negative Y direction. Furthermore, if σ is defined as

$$\sigma/\sigma_0 = p\rho/\rho_0(p + \rho w^2) \tag{50}$$

Eq. (48) gives:

$$\frac{\sigma}{\sigma_0}R = -\frac{\partial\psi}{\partial X} = -\frac{u\rho/\rho_0}{(p + \rho w^2)/p_0}$$

$$\frac{\sigma}{\sigma_0}S = \frac{\partial\psi}{\partial Y} = \frac{v\rho/\rho_0}{(p + \rho w^2)/p_0} \tag{51}$$

Comparing Eq. (51) with Eq. (12), it is evident that σ can be considered as the density of a fluid in the XY plane. Therefore, there exists a complete reciprocity between the xy plane and the XY plane, as shown by Bateman.[6]

Transformation Starting with Incompressible Flow

So far the relations obtained are general, *i.e.*, they apply to fluids of arbitrary properties. However, since only the flow of incompressible fluids is well known, it would be interesting to find the properties of the fluid in the XY plane if the fluid in the xy plane is incompressible. If the fluid in the xy plane is incompressible, then $\rho/\rho_0 = 1$, and the Bernoulli theorem gives:

$$(p + {}^1/_2\rho w^2)/p_0 = 1 \tag{52}$$

Let P denote the pressure in the XY plane, and Q^2 denote $R^2 + S^2$; then Eq. (3), the generalized Bernoulli theorem, gives:

$$^1/_2 Q^2 + \int dp/\sigma = \text{constant} \tag{53}$$

In view of Eqs. (49), (50), and (52), Eq. (53) can be written in the following form:

$$\frac{1}{\sigma_0}\int \frac{d\left(\frac{\sigma}{\sigma_0}\right)}{\frac{\sigma}{\sigma_0}}\,\frac{dP}{d\left(\frac{\sigma}{\sigma_0}\right)} + \frac{p_0}{\rho_0}\left(\frac{1}{4\frac{\sigma^2}{\sigma_0^2}} - \frac{1}{4}\right) = \text{constant} \tag{54}$$

By differentiating Eq. (54) with respect to σ/σ_0, multiplying the resulting expression by σ/σ_0 and then integrating with respect to σ/σ_0, the following relation connecting the pressure P and the density σ for the fluid in the XY plane is obtained:

$$P = C - \frac{1}{2}\frac{p_0}{\rho_0}\sigma_0^2\frac{1}{\sigma} \tag{55}$$

where C is the integration constant. Comparing Eq. (55) with the approximate adiabatic relation Eq. (2), also noting Eq. (6), it is evident that Eqs. (55) and (2) are identical, if

$$\frac{1}{2}\frac{p_0}{\rho_0} = A_0^2 = A_1^2\left[1 - \left(\frac{Q_1}{A_1}\right)^2\right] \tag{56}$$

and

$$C = P_1 + \frac{1}{2}\frac{p_0}{\rho_0}\sigma_0^2\frac{1}{\sigma_1}$$

In Eq. (56) A is the velocity of sound in the XY plane, and the subscript 1 refers to the conditions in the undisturbed flow. Hence, Q_1/A_1 is Mach Number of the undisturbed flow.

By using Eqs. (52) and (49) the components of velocity in the XY plane can be expressed as

$$\frac{R}{Q_1} = -\frac{v}{w_1}\,\frac{1 - \frac{1}{2}\frac{\rho_0}{p_0}w_1^2}{1 - \frac{1}{2}\frac{\rho_0}{p_0}w_1^2\left(\frac{w}{w_1}\right)^2}$$

$$\frac{S}{Q_1} = \frac{u}{w_1}\,\frac{1 - \frac{1}{2}\frac{\rho_0}{p_0}w_1^2}{1 - \frac{1}{2}\frac{\rho_0}{p_0}w_1^2\left(\frac{w}{w_1}\right)^2} \tag{57}$$

Hence,

$$\frac{Q}{Q_1} = \frac{w}{w_1}\,\frac{1 - \frac{1}{2}\frac{\rho_0}{p_0}w_1^2}{1 - \frac{1}{2}\frac{\rho_0}{p_0}w_1^2\left(\frac{w}{w_1}\right)^2} \tag{58}$$

The relation between w_1 and Q_1 can then be obtained from Eqs. (56) and (57), that is:

$$\frac{1}{2}\frac{\rho_0}{p_0}w_1^2 = \frac{(Q_1/A_1)^2}{\left[1+\sqrt{1-(Q_1/A_1)^2}\right]^2} = \lambda \quad (59)$$

Thus Eq. (58) can be rewritten as:

$$\frac{Q}{Q_1} = \frac{w}{w_1}\frac{1-\lambda}{1-\lambda(w/w_1)^2} \quad (60)$$

Using Eqs. (55), (56) and (50), the pressure coefficient Π in the XY plane, which is defined by $\Pi = (P - P_1) \div {}^1/_2\sigma_1 Q_1^2$, can be expressed as:

$$\Pi = \left(1 + \frac{1}{2}\frac{p_0}{\rho_0}w_1^2\right)\frac{1-(w/w_1)^2}{1-\frac{1}{2}\frac{\rho_0}{p_0}w_1^2(w/w_1)^2}$$

Substituting the value of λ from Eq. (59) in the above expression, the following relation is obtained:

$$\Pi = (1+\lambda)\frac{1-(w/w_1)^2}{1-\lambda(w/w_1)^2} \quad (61)$$

To find the coordinates X and Y in terms of x, y, Eq. (46) must be integrated. It is convenient in this case to use the complex potential of the incompressible flow in the xy plane. If

$$\phi + i\psi = w_1 G(x+iy) = w_1 G(z) \quad (62)$$

then it can be shown with the aid of Eq. (52) that:

$$\bar{Z} = X - iY = i\bar{z} - \frac{1}{2}\frac{\rho_0}{p_0}w_1^2 i\int\left(\frac{dG}{dz}\right)^2 dz$$

where $\bar{z}$ is the complex conjugate of z. Or, writing Z and $\bar{G}$ as the complex conjugates of $\bar{Z}$ and G:

$$e^{i\pi/2}Z = z - \lambda\int\left(\frac{d\bar{G}}{d\bar{z}}\right)^2 d\bar{z} \quad (63)$$

where the factor $e^{i\pi/2}$ will rotate the Z plane through an angle equal to $\pi/2$ to make the directions of undisturbed flow in the Z plane and in the z plane coincide.

Comparing the set of Eqs. (59), (60), (61), and (63) with the previous set of Eqs. (33), (34), (35), and (32), it is evident that the two sets are identical except the change of notation. Therefore, Bateman's transformation does not give any new results as it leads to the same expressions as those obtained by the hodograph method.

Concluding Remarks

It is shown both in Section I and in Section II that starting from any solution of an incompressible fluid over a solid body, a solution of a nearly adiabatic flow over another approximately similar solid body can be calculated. The transformation from incompressible flow to compressible flow changes the shape of the body a small amount as represented by the correction terms in Eqs. (32) and (63). Thus, in order to investigate the effect of compressibility over the same body, it is necessary to use different functions $G(z)$ for different Mach Numbers, as shown by the example given in Section I. This complicates the calculations to some extent, but the amount of labor involved is probably much less than the successive approximations devised by Janzen, Rayleigh, Poggi, and Walther, especially at higher Mach Numbers.

The main difficulty of the method lies in its application to flow involving circulation, *e.g.*, the flow over a lifting airfoil. Then if the ordinary complex potential function $G(z)$ for the incompressible fluid is used, the correction terms in Eqs. (32) and (63) are no longer single-valued functions, that is, they do not return to their original value by increasing the argument of z by 2π. In other words, the boundary in compressible flow is no longer a closed curve. Therefore, in order to study this type of problem, it is necessary to use a function $G(z)$ which does not give a closed boundary in the incompressible flow, but will give a closed boundary in the compressible flow when the correction term is added. The problem is thus more difficult, and requires further study.

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